5.3 Diagonalization

Key idea: Matrices ($n\times n$) with n linearly independent eigenvectors are similar to diagonal matrices. This fact is useful in that computations with diagonal matrices are very simple and fast.

Diagonal matrices are extremely simple to compute with:

Ex/ If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ then $D^2 = DD = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$D^3 = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & 1^3 & 0 \\ 0 & 0 & -2^3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$ and for all k $D^k = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & -2^k \end{bmatrix}$.

Thus, computations with a given matrix of interest A will be greatly assisted if we can translate them to computations for a diagonal matrix D associated to A.

Def: Matrices A and B are similar if there is an invertible matrix P s.t. $A = PBP^{-1}$ (i.e. $AP = PB$).

A is diagonalizable if A is similar to a diagonal matrix D.

Our goal in this section is to determine for a given matrix A if
1) if A is diagonizable.

And if so, 2) an invertible matrix P and diagonal matrix D that witness this: $A = PDP^{-1}$.

Ex/ $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is diagonalizable.

A is similar to $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ by $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$\hookrightarrow P^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$= P \cdot D \cdot P^{-1}$

Notice two things:

1) $$A^2 = (PDP^{-1})(PDP^{-1}) = P\,D\underbrace{(P^{-1}P)}_{I}DP^{-1} = P\,D^2P^{-1}$$

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\Rightarrow A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 5^k - 3^k & 5^k - 3^k \\ 2\cdot 3^k - 2\cdot 5^k & 2\cdot 3^k - 5^k \end{bmatrix}$$

i.e. computation is made much easier

!! 2) $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$. Let $\lambda_1 = 5$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\lambda_2 = 3$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Notice $A\vec{v}_1 = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda_1\vec{v}_1$ so P has eigenvectors of A for columns

$A\vec{v}_2 = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda_2\vec{v}_2$

and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ has eigenvalues for entries (and the order agrees!).

Fact: A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case $A = PDP^{-1}$ for P having linearly independent eigenvectors corresponding to the eigenvalues of A which are the diagonal entries of D.

So to diagonalize A an $n \times n$ matrix:

1) find all eigenvalues of A
2) find n linearly independent eigenvectors of A.
3) Place these vectors in a matrix P
4) Place the corresponding eigenvalues in D (repeating as necessary).

$\hookrightarrow$ 5) $A = PDP^{-1}$

Note: if $\lambda_1 \neq \lambda_2$ then $\{\vec{v}_1, \vec{v}_2\}$ e-vecs for λ_1, λ_2 are linearly independent.

$\Downarrow$

Fact: A is diagonalizable if A has n distinct eigenvalues

Ex/ If possible, diagonalize $A=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

1) $\det(A-\lambda I) = \cdots = -\lambda^3 - 3\lambda^2 + 4$
$= -(\lambda-1)(\lambda+2)^2 \Rightarrow$ eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -2$.

2) We need 3 linearly independent eigenvectors.

$$\begin{bmatrix} A - I & \vec{0} \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & -6 & -3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

one free variable

$\lambda_1 = 1$ has a one dimensional eigenspace: $\text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

so obtain one lin. indep. eigenvector $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$.

$$\begin{bmatrix} A + 2I & \vec{0} \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

two free variables!

3 lin. indep. e.vectors of A

$\lambda_2 = -2$ has a two dimensional eigenspace: $\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

So obtain two lin. indep. eigenvectors $\boxed{\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$.

3) So $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

λ_1 λ_2

4) And $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ so that 5) $A = PDP^{-1}$

Indeed, A is similar to D by P because

To verify we are correct, check $AP = PD$ to avoid finding P^{-1}

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$= P \cdot D$.

Ex) Which of the two matrices are diagonalizable?

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}.$$

eigenvalues: $\det(A - \lambda I) = \cdots$
$= -(\lambda - 1)(\lambda + 2)^2$

$\Rightarrow \quad \lambda_1 = 1, \ \lambda_2 = -2$

2 distinct eigenvalues so λ_2 needs to have two lin. indep. eigenvectors.

eigenspace: $\lambda_1 = 1 \cdots \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$

$\lambda_2 = -2 \cdots \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

So there are only two lin. indep. eigenvectors of A, so A is <u>not</u> diagonalizable.

We will next see how to apply this theory to linear transformations to make those easier to compute and understand.

$\det(B - \lambda I) = (\lambda - 5)(-\lambda)(\lambda + 2)$

$\lambda_1 = 5, \ \lambda_2 = 0, \ \lambda_3 = -2$

$\downarrow$

3 distinct eigenvalues and B is 3×3, so yes!
B <u>is</u> diagonalizable

One more example?

if $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

then $A = PDP^{-1}$ (A is diagonalizable, A is similar to D) where

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$